

From integrable equations to Laurent recurrences

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Abstract

Based on a recursive factorisation technique, first introduced in [43], we show how integrable difference equations give rise to recurrences which possess the Laurent property. We derive non-autonomous Somos- k sequences, with $k = 4, 5$, whose coefficients are periodic functions with period 8 for $k = 4$, and period 7 for $k = 5$, and which possess the Laurent property. We also apply our method to the DTKQ- N equation [7], with $N = 2, 3$, and derive Laurent recurrences with $N + 2$ terms, of order $N + 3$. In the case $N = 3$ the recurrence has periodic coefficients with period 8. We demonstrate that recursive factorisation also provides a proof of the Laurent property.

1 Introduction

A sequence $\{u_n\}_{n=1}^{\infty}$ defined by N initial values $\{u_n\}_{n=1}^N$ and an N th order nonlinear rational recursion,

$$u_{n+N} = R(u_n, u_{n+1}, \dots, u_{n+N-1}), \quad (1)$$

where R is a rational function, is said to have the *Laurent property* if, for all n , u_n is polynomial in the variables $\{u_n^{\pm 1}\}_{n=1}^N$. The property was first introduced by Hickerson to prove the integrality of a sequence called Somos-6, cf. [38]. Indeed, as an immediate consequence of the Laurent property it follows that the sequence obtained by taking $\{u_n = 1\}_{n=1}^N$ is an integer sequence, or, a sequence of polynomials if the rational function R depends on additional parameters. For example, with mentioned initial values the (generalized) Somos-4 recurrence

$$\tau_{n+2}\tau_{n-2} = \alpha\tau_{n+1}\tau_{n-1} + \beta\tau_n^2 \quad (2)$$

provides a sequence of polynomials in two variables α, β .

Equation (2) was derived (in 1982) by Michael Somos as an addition formula for elliptic functions. It is the prototype Laurent recurrence, and it has many beautiful properties. The sequence of numbers that one gets by taking $\alpha = \beta = 1$ is referred to as *the* Somos-4 sequence. Its integrality (and of related sequences) was a great mystery initially [6, 18, 32, 40]. Robinson showed that the i -th and j -th terms of the Somos-4 sequence are relatively prime whenever $|i - j| \leq 4$, and he inferred that that for any given $m \in \mathbb{N}$ the sequence modulo m is periodic [38]. Everest et al. [8] showed that every term beyond the fourth has a primitive divisor, i.e. a prime which does not divide any preceding term. Kanki et al. [29] have proven that all terms of Somos-4 are irreducible Laurent polynomials in their initial values and pairwise co-prime, as Laurent polynomials. A seemingly unnoticed divisibility property for the Somos-4 polynomials was recently found by one of the authors [43]. A so called near-addition formula has been proven in [31]. Somos-4 is closely connected to an elliptic divisibility sequence (EDS) [26, 28, 35, 45], the theory of which recently found application in cryptography [39], and in generating large primes [9]. An explicit solution for τ_n in terms of the Weierstrass elliptic function can be found in [24, 26]. From an integrable

systems viewpoint, the Somos-4 recurrence arises as a bilinearisation of the following QRT map [37],

$$f_{n+1}f_nf_{n-1} = \alpha + \frac{\beta}{f_n}, \quad (3)$$

through the relation

$$f_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \quad (4)$$

which encodes the singular confinement of the QRT map [25, 27]. Furthermore, it is a special case of the Hirota-Miwa equation [21, 33, 46].

A deeper understanding of the Laurent property, for a wide class of recurrences, came with the work of Fomin and Zelevinski [13, 14], and subsequent development [1, 17, 30]. The algebraic combinatorial setting of cluster algebras has had profound impact in diverse areas of mathematics, such as algebraic Lie theory [19], Poisson geometry [20], higher Teichmüller theory [11], the representation theory of quivers and finite-dimensional algebras [5], and integrable systems [16], cf. the cluster algebra portal [12].

In this paper we describe a method to obtain recurrences which possess the Laurent property, such as (2), from equations that are integrable, such as (3), which is different than via a transformation such as (4). Our method is based on a recursive factorisation technique, which was used in [43] to establish polynomial upper bounds on the growth of degrees of rational mappings. Slow growth (= low complexity = vanishing algebraic entropy) is a good indicator (better than singular confinement) of the integrability of a mapping [2, 3, 4, 10, 36, 44].

Given a rational recurrence (1) we homogenise $u_n = a_n/b_n$, which yields a system of recurrences for polynomial sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$. Such a system has two ultra-discrete limits; one gives an upper bound on the growth of the degrees of the polynomials a_n and b_n , and the other yields a lower bound on the multiplicities of their divisors. The degree of u_n is obtained from the degree of a_n (or b_n) minus the degree of the greatest common divisor $g_n = \gcd(a_n, b_n)$. Thus, one has to control the divisors of a_n and b_n . By iterating the system finitely many times (one has to iterate N times, the order of the recurrence) and using the observed factorisation as initial values in the ultra-discrete system for multiplicities, one obtains a lower bound on the multiplicities of divisors [22, 43]. In many cases this lower bound on the multiplicities is sharp. In any case, by recursively defining the next divisor to be the quotient of a term in the sequence after division by the previous divisors, one produces an exact factorisation of the polynomial sequences (although not necessarily into irreducible factors)¹. Substitution of these factorisations, when all but finitely many divisors are common, into the system of recurrences for our polynomial sequences yields a nonlinear rational recurrence for the divisors. Clearly, by definition, the divisors are polynomial. Hence one expects the recurrence to possess the Laurent property.

If one starts with a recurrence (1) that has the Laurent property, what will happen is that all divisors but powers of the initial variables, will be common to both a_n and b_n for all n . This then proves the Laurent property.

In section 2 we show how the QRT-map (3), via recursive factorisation, gives rise to a Somos-4 recurrence of the form (2) but where the coefficients are now functions of the initial values of the QRT-map, $\alpha = \alpha_n(f_1, f_2), \beta = \beta_n(f_1, f_2)$, which satisfy the periodicity conditions $(\alpha, \beta)_{n+p} = (\alpha, \beta)_n$ with $p = 8$. Similarly we show how another QRT-map yields a Somos-5 recurrence where the coefficients are periodic functions with period $p = 7$. In section 3 we follow the same procedure starting with the Somos-4,5 sequences themselves. Surprisingly, or not, they give rise to Somos-4 and Somos-5 recurrences with more general periodic coefficients than those obtained in section 2. Explicitly we have found

$$c_{n+2}c_{n-2} = \alpha_n c_{n+1}c_{n-1} + \beta_n c_n^2, \quad (5)$$

¹If one is after degree growth one now writes the degree of a_n (or b_n) as a convolution of the degrees of the divisors and their multiplicities. Using (the solution to) the degree recurrence one may then obtain a recursion for the degrees of the divisors and, when all but finitely many divisors are common, retrieve an upper bound on the growth of degrees of u_n [22, 43].

with coefficients

$$\alpha_n = \alpha \prod_{i=1}^4 \tau_i^{p_{n-i \bmod 8}}, \quad \beta_n = \beta \prod_{i=1}^4 \tau_i^{q_{n-i \bmod 8}}. \quad (6)$$

where $p = [1, 0, 0, 1, 0, 0, 1, 0]^2$, $q = [0, 0, 1, 0, 1, 0, 0, 2]$, and

$$d_{n+3}d_{n-2} = \gamma_n d_{n+2}d_{n-1} + \delta_n d_n d_{n+1}, \quad (7)$$

with coefficients

$$\gamma_n = \gamma \prod_{i=1}^5 \sigma_i^{r_{n-i \bmod 7}}, \quad \delta_n = \delta \prod_{i=1}^5 \sigma_i^{s_{n-i \bmod 7}}. \quad (8)$$

where $r = [1, 0, 0, 0, 1, 0, 0]$, $s = [0, 0, 1, 0, 0, 1, 1]$. Both equations (5,7) are special cases of the Hirota-Miwa equation

$$h_n h_{n+w} = \alpha_n f_{n+v_1} f_{h+u_1} + \beta_n f_{n+v_2} f_{h+u_2} \quad (9)$$

whose integrability condition

$$\alpha_n \alpha_{n+w} \beta_{n+v_1} \beta_{n+u_1} = \alpha_{n+v_2} \alpha_{n+u_2} \beta_n \beta_{n+w} \quad (10)$$

is equivalent to Laurentness, see [33]. Condition (10) is satisfied for both equations (5) and (7).

In the final section we apply our method to two equations quite unrelated to anything Somos. We consider the first two members of the hierarchy of equations

$$\left(\sum_{k=0}^N u_{n+k} \right) \left(\prod_{l=1}^{N-1} u_{n+l} \right) = \phi. \quad (11)$$

which was introduced in [7], and whose degree growth has been studied in [22]. For $N = 2$ the map is another QRT-map, which we relate to the fifth order Laurent recurrence

$$e_{n+5}e_{n+2}^2e_{n+1} = \phi e_{n+3}^2e_{n+2}^2 - e_{n+4}e_{n+3}^2e_n - e_{n+4}^2e_{n+1}^2. \quad (12)$$

For $N = 3$ we find that the ultra-discretisation of the homogenised system does not describe the multiplicities of the second divisor. Using a number of primes as initial values enables us to iterate the system sufficiently many times and so to formulate a conjecture for these multiplicities. Via recursive factorisation we arrive at the following sixth order periodic Laurent recurrence,

$$\begin{aligned} \epsilon_{n+3}c_{n-1}c_{n-2}c_n^2c_{n+3} &= \frac{\alpha}{\epsilon_{n+1}\epsilon_{n+2}} c_{n-1}c_n^3c_{n+1} - \epsilon_n c_{n+2}c_{n-3}c_{n+1}c_n^2 \\ &\quad - \epsilon_{n+1}c_{n-2}^2c_{n+2}c_{n+1}^2 - \epsilon_{n+2}c_{n-2}c_{n-1}^2c_{n+2}^2, \end{aligned}$$

with $\epsilon_n = u_2^{\zeta_{n \bmod 8}}$ and $\zeta = [0, 1, 0, -1, -1, 2, -1, -1]$.

If one starts with an integrable equation and obtains via recursive factorisation a Laurent recurrence for the divisors of the numerators and denominators, the divisors will depend on both the parameters and the initial values of the integrable equation. For the periodic Somos sequences this dependence is realised in the coefficients from the Laurent recurrence, and we can start the recurrence with unit initial values. Thus the polynomiality of the divisors is completely explained by the Laurentness of the recurrence. For the recurrences we have obtained for the DTKQ equations this is not the case. Here we have to initialise the recurrences with initial values that depend in a specific way on the initial values of the DTKQ equation. Therefore in these cases the Laurentness of the recurrences is not enough to explain the polynomiality of the divisors. One would need a strong Laurent property such as given for Somos-4,5 in [28]. This issue is left open for future research.

²We have $p_1 = 1$.

2 From QRT maps to Somos-4 & 5 recurrences with periodic coefficients

In this section we show how by homogenisation, an ultra-discrete limit and recursive factorisation the QRT-map (3) leads to a special case of periodic Somos-4, equation (5). A similar result for Somos-5 is also given.

2.1 To periodic Somos-4

We substitute $f_n = a_n/b_n$ in (3). This gives

$$\frac{a_{n+1}}{b_{n+1}} = \frac{w_{n+1}b_nb_{n-1}}{a_{n-1}a_n^2},$$

with $w_{n+1} := \alpha a_n + \beta b_n$, from which we obtain a system of recurrences for polynomial sequences $\{a_n\}$ and $\{b_n\}$:

$$a_{n+1} = w_{n+1}b_nb_{n-1} \quad (13)$$

$$b_{n+1} = a_{n-1}a_n^2, \quad (14)$$

which we supplement with initial values $a_1 = f_1$, $a_2 = f_2$, $b_1 = b_2 = 1$. Iterating (13) and (14) three more times give us:

$$\begin{aligned} a_{n+2} &= a_{n-1}a_n^2b_nr_1, \quad b_{n+2} = a_nb_{n-1}^2b_n^2w_{n+1}^2, \\ a_{n+3} &= a_{n-1}a_n^4b_{n-1}^2b_n^3r_2w_{n+1}^2, \quad b_{n+3} = a_{n-1}^2a_n^4b_{n-1}b_n^3w_{n+1}r_1^2, \\ a_{n+4} &= a_{n-1}^3a_n^9b_{n-1}^4b_n^8r_2^2r_3w_{n+1}^4, \quad b_{n+4} = a_{n-1}^3a_n^{10}b_{n-1}^4b_n^7r_2^2w_{n+1}^4, \end{aligned}$$

where $\{r_i\}_{i=1}^3$ are irreducible polynomials in a_{n-1} , b_{n-1} , a_n , b_n , α and β . We observe the following factorisation properties: w_{n+1} does not divide a_{n+2} , it divides b_{n+2} and a_{n+3} with multiplicity 2, it divides b_{n+3} with multiplicity 1, and w_{n+1} is a divisor of both a_{n+4} and b_{n+4} with multiplicity 4. Furthermore, from (13) and (14), we find the following ultra-discrete system of recurrences for multiplicities:

$$m_{n+2}^a \geq \min(m_{n+1}^a + m_n^b + m_{n+1}^b; 2m_{n+1}^b + m_n^b),$$

$$m_{n+2}^b = m_n^a + 2m_{n+1}^a.$$

where $m_i^p(f)$ denotes the multiplicity of a polynomial f in polynomial p_i and we have suppressed the dependence on f . Using the equal sign in the first equation, we will get a lower bound for the multiplicities, which we denote using Euler's fracture typesetting. Thus, we will employ the following system:

$$\mathfrak{m}_{n+2}^a = \min(\mathfrak{m}_{n+1}^a + \mathfrak{m}_n^b + \mathfrak{m}_{n+1}^b; 2\mathfrak{m}_{n+1}^b + \mathfrak{m}_n^b), \quad (15)$$

$$\mathfrak{m}_{n+2}^b = \mathfrak{m}_n^a + 2\mathfrak{m}_{n+1}^a. \quad (16)$$

To get a lower bound for the multiplicity of w_k ($k > 2$) in the sequences $\{a_n\}$ and $\{b_n\}$, we solve (15) and (16) with the following initial values: $\mathfrak{m}_{k+1}^a = 0$, $\mathfrak{m}_{k+1}^b = 2$, $\mathfrak{m}_{k+2}^a = 2$, $\mathfrak{m}_{k+2}^b = 1$ and $\mathfrak{m}_{k+3}^a = \mathfrak{m}_{k+3}^b = 4$. We find, for $n \geq k+3$, that $\mathfrak{m}_n^a(w_k) = \mathfrak{m}_n^b(w_k) = \mathfrak{m}_{n-k}$, where

$$\mathfrak{m}_1 = 0, \quad \mathfrak{m}_2 = 2, \quad \mathfrak{m}_{n+1} = 2\mathfrak{m}_n + \mathfrak{m}_{n-1}.$$

This can be seen by taking $\mathfrak{m}_k^a = \mathfrak{m}_k^b$ in the right hand sides of (15) and (16). One finds equality and hence $\mathfrak{m}_{n+2}^a = \mathfrak{m}_{n+2}^b$. We define sequences $\{\mathfrak{m}_n^a(c_i)\}_{n=1}^\infty$ and $\{\mathfrak{m}_n^b(c_i)\}_{n=1}^\infty$, for $i \in \{1, 2\}$, by (15) and (16) and initial values $\mathfrak{m}_j^a(c_i) = \delta_{ij}$ and $\mathfrak{m}_j^b(c_i) = 0$.

The polynomials a_n and b_n can be expressed in terms of a sequence $\{c_k\}_{k=1}^\infty$, of polynomials in a_1, a_2, α and β . Each polynomial c_n is defined as the quotient of a_n after division by powers of $c_{i < n}$, as follows,

$$a_n = \begin{cases} c_n & \text{if } n \leq 3, \\ c_1 c_2^2 c_4 & \text{if } n = 4, \\ c_1^{\mathfrak{m}_n^a(c_1)} c_2^{\mathfrak{m}_n^a(c_2)} (\prod_{i=3}^{n-3} c_i^{\mathfrak{m}_n^a(c_i)}) c_{n-2}^2 c_n & \text{if } n > 4. \end{cases} \quad (17)$$

It is clear that c_n is polynomial because $c_i | w_i$ for all $i > 4$ and hence $\mathfrak{m}_n^a(c_i) \geq \mathfrak{m}_n^a(w_i)$. We know that $\prod_{i=1}^n c_i^{\mathfrak{m}_n^b(c_i)} | b_n$. Taking b_n to be given by

$$b_n = \begin{cases} 1 & \text{if } n \leq 2, \\ c_{n-2} c_{n-1}^2 & \text{if } n \in \{3, 4\}, \\ c_1^{\mathfrak{m}_n^b(c_1)} c_2^{\mathfrak{m}_n^b(c_2)} (\prod_{i=3}^{n-3} c_i^{\mathfrak{m}_n^b(c_i)}) c_{n-2} c_{n-1}^2, & \text{if } n > 4, \end{cases} \quad (18)$$

we can verify equation (46) is satisfied. Thus, defining $g_n = \gcd(a_n, b_n)$ to be the greatest common divisor of a_n and b_n , we get

$$g_n = \prod_{i=1}^n c_i^{\mathfrak{m}_n^g(c_i)} = c_1^{\mathfrak{m}_n^g(c_1)} c_2^{\mathfrak{m}_n^g(c_2)} (\prod_{i=3}^{n-3} c_i^{\mathfrak{m}_n^g(c_i)}) c_{n-2}, \quad (19)$$

where $\mathfrak{m}_n^g(c_i) = \min(\mathfrak{m}_n^a(c_i), \mathfrak{m}_n^b(c_i))$. Note, from $\frac{b_n}{g_n} = c_1^{\mathfrak{m}_n^b(c_1) - \mathfrak{m}_n^g(c_1)} c_2^{\mathfrak{m}_n^b(c_2) - \mathfrak{m}_n^g(c_2)} c_{n-1}^2$, it can be seen that the map (3) does not possess the Laurent property.

We now consider the lower bounds for the multiplicities of c_1, c_2 in a_n and b_n and observe the following differences are periodic.

Lemma 1. *We have:*

$$\mathfrak{m}_k^a(c_i) - \mathfrak{m}_k^b(c_i) = \begin{cases} v_k \bmod 8 & \text{if } i = 1, \\ v_{k-3} \bmod 8 & \text{if } i = 2, \end{cases}$$

where $v = [1, 0, -1, 1, -1, 0, 1, -2]$.

Proof. By induction. We only give the case $k \equiv 1 \bmod 8$ for c_1 . For $k = 1$, it is evident. Suppose the claim is true for $i < k$. Then, we have $\mathfrak{m}_{k-1}^a(c_1) = \mathfrak{m}_{k-1}^b(c_1) - 2$ and $\mathfrak{m}_{k-2}^a(c_1) = \mathfrak{m}_{k-2}^b(c_1) + 1$, hence

$$\begin{aligned} \mathfrak{m}_k^a(c_1) &= \min(\mathfrak{m}_{k-1}^a(c_1) + \mathfrak{m}_{k-2}^b(c_1) + \mathfrak{m}_{k-1}^b(c_1), 2\mathfrak{m}_{k-1}^b(c_1) + \mathfrak{m}_{k-2}^b(c_1)) \\ &= 2\mathfrak{m}_{k-1}^b(c_1) + \mathfrak{m}_{k-2}^b(c_1) - 2, \text{ and} \\ \mathfrak{m}_k^b(c_1) &= \mathfrak{m}_{k-2}^a(c_1) + 2\mathfrak{m}_{k-1}^a(c_1) \\ &= 2\mathfrak{m}_{k-1}^b(c_1) + \mathfrak{m}_{k-2}^b(c_1) - 3. \end{aligned}$$

So $\mathfrak{m}_k^a(c_1) - \mathfrak{m}_k^b(c_1) = 1$. By applying the same technique for other cases ($k \equiv 2, \dots, 8 \bmod 8$) and for c_2 , the lemma is proven. \square

From (17), (18), (19), it follows that

$$\alpha_n := \frac{\alpha a_n}{c_n c_{n-2} g_n} \quad \text{and} \quad \beta_n := \frac{\beta b_n}{c_{n-1}^2 g_n} \quad (20)$$

are polynomials in c_1, c_2 . As a Corollary to Lemma 1, it follows that α_n and β_n are periodic sequences of period 8.

Corollary 2. *We have:*

$$\alpha_n = \alpha c_1^{p_n \bmod 8} c_2^{p_{n-3} \bmod 8} \quad \text{and} \quad \beta_n = \beta c_1^{q_n \bmod 8} c_2^{q_{n-3} \bmod 8}, \quad (21)$$

with $p = [1, 0, 0, 1, 0, 0, 1, 0]$ and $q = [0, 0, 1, 0, 1, 0, 0, 2]$.

Proof. We have:

$$\alpha_n = \frac{\alpha a_n}{g_n c_n c_{n-2}} = \alpha c_1^{\mathfrak{m}_n^a(c_1) - \mathfrak{m}_n^g(c_1)} c_2^{\mathfrak{m}_n^a(c_2) - \mathfrak{m}_n^g(c_2)},$$

where

$$\mathfrak{m}_n^a - \mathfrak{m}_n^g = \begin{cases} \mathfrak{m}_n^a - \mathfrak{m}_n^b & \text{if } \mathfrak{m}_n^a - \mathfrak{m}_n^b > 0, \\ 0 & \text{if } \mathfrak{m}_n^a - \mathfrak{m}_n^b \leq 0. \end{cases}$$

Therefore,

$$\mathfrak{m}_n^a(c_i) - \mathfrak{m}_n^g(c_i) = \begin{cases} p_n \bmod 8 & \text{if } i = 1, \\ p_{n-3} \bmod 8 & \text{if } i = 2, \end{cases}$$

where $p_k = \max(0, v_k)$. Similarly, we have:

$$\beta_n = \frac{\beta b_n}{g_n c_{n-1}^2} = \beta c_1^{\mathfrak{m}_n^b(c_1) - \mathfrak{m}_n^g(c_1)} c_2^{\mathfrak{m}_n^b(c_2) - \mathfrak{m}_n^g(c_2)},$$

where

$$\mathfrak{m}_n^b(c_i) - \mathfrak{m}_n^g(c_i) = \begin{cases} q_n \bmod 8 & \text{if } i = 1, \\ q_{n-3} \bmod 8 & \text{if } i = 2, \end{cases}$$

with $q_k = \max(0, -v_k)$. □

Theorem 3. *The polynomials c_n , as defined by (17), satisfy*

$$c_3 = \alpha c_2 + \beta, \quad c_4 = \alpha c_3 + \beta c_1 c_2^2, \quad c_5 = \alpha c_1 c_2 c_4 + \beta c_3^2, \quad c_6 = \alpha c_5 c_3 + \beta c_1 c_4^2, \quad (22)$$

and, for $n \geq 6$,

$$c_{n-3} c_{n+1} = \alpha_n c_n c_{n-2} + \beta_n c_{n-1}^2. \quad (23)$$

Proof. Using equations (13) and (14), initial values and (17), we find:

$$c_3 = a_3 = (\alpha a_2 + \beta b_2) b_1 b_2 = (\alpha c_2 + \beta).$$

Furthermore,

$$c_4 = \frac{a_4}{c_1^{\mathfrak{m}_4^a(c_1)} c_2^{\mathfrak{m}_4^a(c_2)}} = \frac{(\alpha a_3 + \beta b_3) b_2 b_3}{c_1 c_2^2} = \alpha c_3 + \beta c_1 c_2^2,$$

as $b_3 = c_1 c_2^2$, $\mathfrak{m}_4^a(c_1) = 1$ and $\mathfrak{m}_4^a(c_2) = 2$. Similarly, the formulae for c_5 and c_6 are obtained. Solving equations (20) for a_n and b_n and substituting in equation (13), we find:

$$\frac{\alpha_{n+1} c_{n+1} c_{n-1} g_{n+1}}{\alpha} = (\alpha_n c_n c_{n-2} g_n + \beta_n c_{n-1}^2 g_n) \frac{\beta_{n-1} \beta_n c_{n-1}^2 c_{n-2}^2 g_{n-1} g_n}{\beta^2}.$$

Thus,

$$c_{n-3} c_{n+1} = Z_n (\alpha_n c_n c_{n-2} + \beta_n c_{n-1}^2),$$

with

$$Z_n = \frac{\beta_{n-1} \beta_n}{\beta^2} \frac{\alpha}{\alpha_{n+1}} \frac{g_{n-1} g_n^2}{g_{n+1}} c_{n-1} c_{n-2}^2 c_{n-3}.$$

Substituting in equation (14) gives us:

$$g_{n+1} = \frac{\beta}{\beta_{n+1}} \frac{\alpha_n^2 \alpha_{n-1}}{\alpha^3} g_{n-1} g_n^2 c_{n-1} c_{n-2}^2 c_{n-3},$$

which we use to simplify

$$Z_n = \frac{\alpha^4 \beta_{n-1} \beta_n \beta_{n+1}}{\alpha_{n-1} \alpha_n^2 \alpha_{n+1} \beta^3} = 1$$

as $q_{n-1} + q_n + q_{n+1} = p_{n-1} + 2p_n + p_{n+1}$. □

The fact that the sequence $\{c_n\}_{n=1}^\infty$, with special initial values given by (22) and generated by the rational recurrence (23), is a polynomial sequence is curious. First of all, it follows from the definition of c_n given by (17) which is based on factorization properties of the QRT map (3). But there is a second explanation. When we express the coefficients, cf. Lemma 2, in terms of the initial values of the QRT-map (3), $c_1 = a_1 = f_1$ and $c_2 = a_2 = f_2$, i.e.

$$\alpha_n^f = \begin{cases} \alpha f_1 & \text{if } n \equiv 1 \pmod{8}, \\ \alpha f_2 & \text{if } n \equiv 2 \pmod{8}, \\ \alpha f_1 f_2 & \text{if } n \equiv 4, 7 \pmod{8}, \\ \alpha & \text{if } n \equiv 3, 5, 6, 8 \pmod{8}, \end{cases} \quad \beta_n^f = \begin{cases} \beta & \text{if } n \equiv 1, 2, 4, 7 \pmod{8}, \\ \beta f_1 f_2^2 & \text{if } n \equiv 3 \pmod{8}, \\ \beta f_1 & \text{if } n \equiv 5 \pmod{8}, \\ \beta f_2 & \text{if } n \equiv 6 \pmod{8}, \\ \beta f_1^2 f_2 & \text{if } n \equiv 8 \pmod{8}, \end{cases} \quad (24)$$

and supplement the recurrence

$$c_{n-3}c_{n+1} = \alpha_n^f c_n c_{n-2} + \beta_n^f c_n^2. \quad (25)$$

with initial values $c_i = 1$ for $i \in \{-1, 0, 1, 2\}$ we find the following expressions

$$c_3 = \alpha f_2 + \beta, \quad c_4 = \alpha c_3 + \beta f_1 f_2^2, \quad c_5 = \alpha f_1 f_2 c_4 + \beta_3^2, \quad c_6 = \alpha c_5 c_3 + \beta f_1 c_4^2,$$

which agree with (22). Therefore, the fact that the sequence consist of polynomials can also be explained by the Laurent property of (25), which we establish in section 3.3.

2.2 To periodic Somos-5

We will now show how the QRT-map

$$h_{n+1}h_nh_{n-1} = \gamma h_n + \delta, \quad (26)$$

which is related to Somos-5,

$$\sigma_{n+3}\sigma_{n-2} = \gamma\sigma_{n+2}\sigma_{n-1} + \delta\sigma_{n+1}\sigma_n, \quad (27)$$

via the transformation, cf. [26],

$$h_n = \frac{\sigma_{n+2}\sigma_{n-1}}{\sigma_{n+1}\sigma_n}, \quad (28)$$

leads to a special case of periodic Somos-5, equation (7). Substituting $h_n = a_n/b_n$, the homogenised system for numerators and denominators is given by:

$$a_{n+1} = w_{n+1}b_{n-1} \quad (29)$$

$$b_{n+1} = a_n a_{n-1}, \quad (30)$$

where $w_{n+1} := \gamma a_n + \delta b_n$. We take $\{b_i = 1\}_{i=1}^2$, so that $\{a_i = h_i\}_{i=1}^2$. Iterating (29), (30), we find:

$$\begin{aligned} a_{n+2} &= b_n s_1, \quad b_{n+2} = w_{n+1} b_{n-1} a_n, \\ a_{n+3} &= a_n a_{n-1} s_2, \quad b_{n+3} = b_n b_{n-1} s_1 w_{n+1}, \\ a_{n+4} &= a_n b_{n-1} s_3 w_{n+1}, \quad b_{n+4} = a_n a_{n-1} b_n s_1 s_2, \\ a_{n+5} &= a_n b_{n-1} b_n s_1 s_4 w_{n+1}, \quad b_{n+5} = a_{n-1} a_n^2 b_{n-1} s_2 s_3 w_{n+1}, \end{aligned}$$

where $\{s_i\}_1^4$ are irreducible polynomials in $\{a_{n+i}, b_{n+i}\}_{i=-1}^0$, δ , and γ . In addition, from (29) and (30), the ultra-discrete system of recurrences for a lower bound on the multiplicities is:

$$\mathfrak{m}_{n+1}^a = \min(\mathfrak{m}_n^a + \mathfrak{m}_{n-1}^b; \mathfrak{m}_n^b + \mathfrak{m}_{n-1}^a), \quad (31)$$

$$\mathfrak{m}_{n+1}^b = \mathfrak{m}_n^a + \mathfrak{m}_{n-1}^a. \quad (32)$$

To get a lower bound for the multiplicity of w_k ($k > 3$) in the sequences $\{a_n\}$ and $\{b_n\}$, we solve (31) and (32) with the following initial values: $\mathbf{m}_{k+1}^a = \mathbf{m}_{k+2}^a = \mathbf{m}_{k+3}^b = 0$ and $\mathbf{m}_{k+1}^b = \mathbf{m}_{k+2}^b = \mathbf{m}_{k+3}^a = \mathbf{m}_{k+4}^a = \mathbf{m}_{k+4}^b = 1$. We find, for all $n \geq k+4$, that $\mathbf{m}_n^a(w_k) = \mathbf{m}_n^b(w_k) = \mathbf{m}_{n-k-3}$ where

$$\mathbf{m}_1 = 1, \mathbf{m}_2 = 2, \mathbf{m}_{n+2} = \mathbf{m}_{n+1} + \mathbf{m}_n.$$

For $i \in \{1, 2\}$ we define sequences $\mathbf{m}_n^a(d_i)$ and $\mathbf{m}_n^b(d_i)$ by (31) and (32) and the initial values $\mathbf{m}_j^a(d_i) = \delta_{ij}$ and $\mathbf{m}_j^b(d_i) = 0$. Then, a polynomial sequence $\{d_n\}_{n=1}^\infty$ is defined by

$$a_n = \begin{cases} d_n & \text{if } n \leq 4, \\ d_1 d_2 d_5 & \text{if } n = 5, \\ (\prod_{i=1}^2 d_i^{\mathbf{m}_n^a(d_i)}) (\prod_{i=3}^{n-4} d_i^{\mathbf{m}_{n-i-3}}) d_{n-3} d_n, & \text{if } n > 5, \end{cases} \quad (33)$$

and we have

$$b_n = \begin{cases} 1 & \text{if } n \leq 2, \\ d_{n-2} d_{n-1} & \text{if } n \in \{3, 4\}, \\ (\prod_{i=1}^2 d_i^{\mathbf{m}_n^b(d_i)}) (\prod_{i=3}^{n-4} d_i^{\mathbf{m}_{n-i-3}}) d_{n-2} d_{n-1} & \text{if } n \geq 5. \end{cases} \quad (34)$$

As before, the difference between the multiplicities of d_1 and d_2 is periodic. We have:

$$\mathbf{m}_k^a(d_i) - \mathbf{m}_k^b(d_i) = \begin{cases} w_k \bmod 7 & \text{if } i = 1, \\ w_{k-4} \bmod 7 & \text{if } i = 2, \end{cases}$$

where $w = [1, 0, -1, 0, 1, -1, -1]$, which can be proven by induction as was done in the proof of Lemma 1. From this, it follows that in terms of the initial values of the map (26), h_1 and h_2 , we have

$$\gamma_n^h := \frac{\gamma a_n}{d_n d_{n-3} g_n} = \gamma h_1^{r_n \bmod 7} h_2^{r_n - 4 \bmod 7} \text{ and } \delta_n^h := \frac{\delta b_n}{d_{n-1} d_{n-2} g_n} = \delta h_1^{s_n \bmod 7} h_2^{s_n - 4 \bmod 7}, \quad (35)$$

where

$$r_k = \max(0, w_k) = [1, 0, 0, 0, 1, 0, 0] \text{ and } s_k = \max(0, -w_k) = [0, 0, 1, 0, 0, 1, 1]. \quad (36)$$

Solving (35) for a_n and b_n in terms of γ_n^h and δ_n^h and substituting into (29) and (30), we find the following recursion relations.

Theorem 4. *The sequence $\{d_n\}_{n=1}^\infty$, defined by (33), satisfies $d_1 = h_1$, $d_2 = h_2$, $d_3 = \gamma h_2 + \delta$,*

$$d_4 = \gamma d_3 + \delta h_1 h_2, \quad d_5 = \gamma d_4 + \delta h_2 d_3, \quad d_6 = \gamma h_1 h_2 d_5 + \delta d_3 d_4, \quad d_7 = \gamma d_3 d_6 + \delta h_1 d_4 d_5, \quad (37)$$

and, for all $n \geq 8$,

$$d_{n-4} d_{n+1} = \gamma_n^h d_n d_{n-3} + \delta_n^h d_{n-1} d_{n-2}. \quad (38)$$

We note that (37) are obtained from (38) by taking initial values $d_i = 1$ for $i \in \{-2, -1, 0, 1, 2\}$.

Again, the fact that $\{d_n\}$ is a sequence of polynomials is therefore also explained by the Laurent property of (38), see section 3.3.

Finally, we'd like to mention that the third order mapping [26, Equation 2.9],

$$u_{n+2} u_n^2 u_{n+1}^2 u_{n-1} = \gamma u_n u_{n+1} + \delta, \quad (39)$$

which is related to Somos-5 via

$$u_n = \frac{\sigma_{n+1} \sigma_{n-1}}{\sigma_n^2},$$

can be recursively factorised as $u_n = a_n/b_n$ with

$$a_n = \begin{cases} d_n & \text{if } n \leq 4, \\ d_1 d_2^2 d_3^3 d_5 & \text{if } n = 5, \\ d_1^{\mathbf{m}_n^a(d_1)} d_2^{\mathbf{m}_n^a(d_2)} d_3^{\mathbf{m}_n^a(d_3)} (\prod_{i=4}^{n-3} d_i^{\mathbf{m}_{n-i}}) d_{n-2}^3 d_n, & \text{if } n > 5, \end{cases} \quad (40)$$

and

$$b_n = \begin{cases} 1 & \text{if } n \leq 3, \\ d_{n-3} d_{n-2}^2 d_{n-1}^2 & \text{if } n \in \{4, 5\}, \\ d_1^{\mathbf{m}_n^b(d_1)} d_2^{\mathbf{m}_n^b(d_2)} d_3^{\mathbf{m}_n^b(d_3)} (\prod_{i=4}^{n-3} d_i^{\mathbf{m}_{n-i}}) d_{n-2}^2 d_{n-1}^2, & \text{if } n > 5, \end{cases} \quad (41)$$

where

$$\mathbf{m}_1 = 0, \mathbf{m}_2 = 3, \mathbf{m}_3 = 7, \mathbf{m}_{n+2} = 2\mathbf{m}_{n+1} + 2\mathbf{m}_n + \mathbf{m}_{n-1},$$

and, for $i \in \{1, 2, 3\}$, $\{\mathbf{m}_n^a(d_i)\}$ and $\{\mathbf{m}_n^b(d_i)\}$ are defined by initial values $\{\mathbf{m}_j^a(d_i) = \delta_{ij}, \mathbf{m}_j^b(d_i) = 0\}_{j=1}^3$, and

$$\mathbf{m}_{n+2}^a = \min(\mathbf{m}_n^a + \mathbf{m}_{n+1}^a + \mathbf{m}_n^b + \mathbf{m}_{n+1}^b + \mathbf{m}_{n-1}^b; 2\mathbf{m}_n^b + 2\mathbf{m}_{n+1}^b + \mathbf{m}_{n-1}^b), \quad (42)$$

$$\mathbf{m}_{n+2}^b = 2\mathbf{m}_n^a + 2\mathbf{m}_{n+1}^a + \mathbf{m}_{n-1}^a. \quad (43)$$

Here, the differences between the multiplicities of d_1 , d_2 , and d_3 are periodic sequences with period 14. We have:

$$\mathbf{m}_k^a(d_i) - \mathbf{m}_k^b(d_i) = \begin{cases} h_{k \bmod 14} & \text{if } i = 1, \\ h_{k \bmod 14} + h_{k+3 \bmod 14} & \text{if } i = 2, \\ h_{k-4 \bmod 14} & \text{if } i = 3, \end{cases}$$

where $h = [1, 0, 0, -1, 1, 0, -1, 0, 1, -1, 0, 0, 1, -2]$, from which it follows that $\phi_n := \frac{a_n}{d_n d_{n-2} g_n}$ and $\psi_n := \frac{b_n}{d_{n-1}^2 g_n}$ are periodic with period 14. However, the coefficients of the periodic Somos-5 recurrence for the sequence $\{d_n\}$ defined by (40),

$$d_{n-3} d_{n+2} = \gamma_{n+2}^u d_{n-2} d_{n+1} + \delta_{n+2}^u d_n d_{n-1}, \quad (44)$$

turn out to have period 7,

$$\begin{aligned} \gamma_{n+2}^u &= \gamma \frac{\psi_{n-1} \psi_n \psi_{n+1} \psi_{n+2}}{\phi_{n-1} \phi_n \phi_{n+1} \phi_{n+2}} = \gamma u_1^{r_n \bmod 7} u_2^{r_n \bmod 7 + r_{n-3} \bmod 7} u_3^{r_{n-3} \bmod 7}, \\ \delta_{n+2}^u &= \delta \frac{\psi_{n-1} \psi_n^2 \psi_{n+1}^2 \psi_{n+2}}{\phi_{n-1} \phi_n^2 \phi_{n+1}^2 \phi_{n+2}} = \delta u_1^{s_n \bmod 7} u_2^{s_n \bmod 7 + r_{n-3} \bmod 7} u_3^{s_{n-3} \bmod 7}, \end{aligned}$$

with r, s as before. Thus, equation (44) sits inside the periodic Somos-5 family mentioned in the introduction, equation (7).

3 From Somos-4 & 5 recurrences to Somos-4 & 5 recurrences with periodic coefficients

In this section we derive the Somos sequences with periodic coefficients mentioned in the introduction, which are more general than the ones obtained from QRT-maps in the previous section.

3.1 Periodic Somos-4

By taking $\tau_n = a_n/b_n$ in Somos-4 we find the system of recurrences for polynomial sequences $\{a_n\}$ and $\{b_n\}$:

$$a_{n+2} = w_{n+2} b_{n-2} \quad (45)$$

$$b_{n+2} = b_{n+1} b_n^2 b_{n-1} a_{n-2}, \quad (46)$$

with $w_{n+2} := \alpha a_{n+1} b_n^2 a_{n-1} + \beta b_{n+1} a_n^2 b_{n-1}$. Taking $\{b_i = 1\}_{i=1}^4$, we have $\{a_i = \tau_i\}_{i=1}^4$. From (45) and (46), we find the following ultra-discrete system of recurrences for a lower bound on multiplicities:

$$\mathbf{m}_{n+2}^a = \min(\mathbf{m}_{n+1}^a + 2\mathbf{m}_n^b + \mathbf{m}_{n-1}^a + \mathbf{m}_{n-2}^b; \mathbf{m}_{n+1}^b + 2\mathbf{m}_n^a + \mathbf{m}_{n-1}^b + \mathbf{m}_{n-2}^a), \quad (47)$$

$$\mathbf{m}_{n+2}^b = \mathbf{m}_{n+1}^b + 2\mathbf{m}_n^b + \mathbf{m}_{n-1}^b + \mathbf{m}_{n-2}^a. \quad (48)$$

Iterating the recurrences (45), (46) four times gives us

$$\begin{aligned} a_{n+3} &= b_{n-1} b_{n+1} p_1, \quad b_{n+3} = a_{n-2} a_{n-1} b_{n-1} b_n^3 b_{n+1}^3, \\ a_{n+4} &= a_{n-2} b_{n-1} b_n^4 b_{n+1}^3 p_2, \quad b_{n+4} = a_{n-2}^3 a_{n-1} a_n b_{n-1}^3 b_n^7 b_{n+1}^6, \\ a_{n+5} &= a_{n-2}^3 a_{n-1} b_{n-1}^3 b_n^9 b_{n+1}^{10} p_3, \quad b_{n+5} = a_{n-2}^6 a_{n-1}^3 a_n a_{n+1} b_{n-1}^6 b_n^{15} b_{n+1}^{13}, \\ a_{n+6} &= a_{n-2}^{10} a_{n-1}^3 a_n b_{n-1}^{10} b_n^{25} b_{n+1}^{23} w_{n+2} p_4, \quad b_{n+6} = a_{n-2}^{13} a_{n-1}^6 a_n^3 a_{n+1} b_{n-2}^{13} b_n^{32} b_{n+1}^{28} w_{n+2}, \end{aligned}$$

where p_1, p_2, p_3, p_4 are irreducible polynomials in $\{a_{n+i}, b_{n+i}\}_{i=-2}^1$, α and β . Thus, considering a lower bound for the multiplicity of w_k ($k > 4$) in the sequences $\{a_n\}$ and $\{b_n\}$, we solve (47) and (48) with the following initial values: $\mathbf{m}_{k+i}^a = \mathbf{m}_{k+i}^b = 0$, where $i \in \{1, 2, 3\}$ and $\mathbf{m}_{k+4}^a = \mathbf{m}_{k+4}^b = 1$. We find, for $n \geq k + 1$,

$$\mathbf{m}_n^a(w_k) = \mathbf{m}_n^b(w_k) = \mathbf{m}_{n-k},$$

where the integer sequence $\{\mathbf{m}_n\}$ is defined by

$$\mathbf{m}_{n+2} = \mathbf{m}_{n+1} + 2\mathbf{m}_n + \mathbf{m}_{n-1} + \mathbf{m}_{n-2},$$

and $\mathbf{m}_1 = \mathbf{m}_2 = \mathbf{m}_3 = \mathbf{m}_4 - 1 = 0$. We define sequences $\{\mathbf{m}_n^a(c_i)\}$ and $\{\mathbf{m}_n^b(c_i)\}$, for $i \in \{1, 2, 3, 4\}$, by (47) and (48) and the initial values $\{\mathbf{m}_j^a(c_i) = \delta_{ij}, \mathbf{m}_j^b(c_i) = 0\}_{i,j=1}^4$. Next, polynomials c_n are defined as a quotient of a_n , with $n > 4$, by

$$a_n = \left(\prod_{i=1}^4 c_i^{\mathbf{m}_n^a(c_i)} \right) \left(\prod_{i=5}^{n-1} c_i^{\mathbf{m}_{n-i}} \right) c_n, \quad (49)$$

and b_n can be expressed as

$$b_n = \left(\prod_{i=1}^4 c_i^{\mathbf{m}_n^b(c_i)} \right) \prod_{i=5}^{n-1} c_i^{\mathbf{m}_{n-i}}.$$

Note that

$$g_n = \left(\prod_{i=1}^4 c_i^{\min(\mathbf{m}_n^a(c_i), \mathbf{m}_n^b(c_i))} \right) \prod_{i=5}^{n-1} c_i^{\mathbf{m}_{n-i}} \quad \text{and} \quad \frac{b_n}{g_n} = \prod_{i=1}^4 c_i^{\mathbf{m}_n^b(c_i) - \min(\mathbf{m}_n^a(c_i), \mathbf{m}_n^b(c_i))}, \quad (50)$$

which shows that Somos-4 possesses the Laurent property. We next study the multiplicities of the divisors $\{c_i\}_{i=1}^4$. But first, let us define the ultra-discrete Somos-4 recurrence

$$r_{n+4} = -r_n + \max(r_{n+3} + r_{n+1}, 2r_{n+2}), \quad (51)$$

for which we take initial values $r_1 = -1, r_2 = r_3 = r_4 = 0$, cf. [15, Example 3.6]. The second difference of this sequence,

$$x_k = r_{k+2} - 2r_{k+1} + r_k, \quad (52)$$

is a periodic sequence of order 8. This follows by iteration of $x_{k+2} + 2x_{k+1} + x_k = \max(x_{k+1}, 0)$, which itself is an ultra-discrete version of the QRT-map (3), cf. [34].

Lemma 5. *For all $1 \leq i \leq 4$, we have: $\mathbf{m}_n^b(c_i) - \mathbf{m}_n^a(c_i) = r_{n-i+1}$.*

Proof. It is enough to proof the lemma for $i = 1$, because $\mathbf{m}_n^a(c_i) = \mathbf{m}_{n-1}^a(c_{i-1})$ and $\mathbf{m}_n^b(c_i) = \mathbf{m}_{n-1}^b(c_{i-1})$ for $i = \{2, 3, 4\}$ and $n > 1$. For brevity we omit the dependence of \mathbf{m}_k^a and \mathbf{m}_k^b on c_1 . From initial values, we see that $\mathbf{m}_n^b - \mathbf{m}_n^a = r_n$ for $1 \leq n \leq 4$. We suppose the lemma is true for all $n \leq k$ and demonstrate it is then also true in the case $n = k + 1$. We have

$$\mathbf{m}_{k+1}^a = \min(\mathbf{m}_k^a + 2\mathbf{m}_{k-1}^b + \mathbf{m}_{k-2}^a + \mathbf{m}_{k-3}^b; \mathbf{m}_k^b + 2\mathbf{m}_{k-1}^a + \mathbf{m}_{k-2}^b + \mathbf{m}_{k-3}^a), \quad (53)$$

$$\mathbf{m}_{k+1}^b = \mathbf{m}_k^b + 2\mathbf{m}_{k-1}^b + \mathbf{m}_{k-2}^b + \mathbf{m}_{k-3}^a. \quad (54)$$

According to the hypothesis, we may replace $\mathbf{m}_l^a = \mathbf{m}_l^b - r_l$ in the right hand sides of (53) and (54). This gives

$$\begin{aligned} \mathbf{m}_{k+1}^a &= \min(\mathbf{m}_k^b + 2\mathbf{m}_{k-1}^b + \mathbf{m}_{k-2}^b + \mathbf{m}_{k-3}^b - r_k - r_{k-2}; \mathbf{m}_k^b + 2\mathbf{m}_{k-1}^b + \mathbf{m}_{k-2}^b + \mathbf{m}_{k-3}^b - 2r_{k-1}) \\ &= \mathbf{m}_k^b + 2\mathbf{m}_{k-1}^b + \mathbf{m}_{k-2}^b + \mathbf{m}_{k-3}^b + \min(-(r_k + r_{k-2}), -2r_{k-1}) \\ &= \mathbf{m}_k^b + 2\mathbf{m}_{k-1}^b + \mathbf{m}_{k-2}^b + \mathbf{m}_{k-3}^b - \max(r_k + r_{k-2}, 2r_{k-1}) \\ \mathbf{m}_{k+1}^b &= \mathbf{m}_k^b + 2\mathbf{m}_{k-1}^b + \mathbf{m}_{k-2}^b + \mathbf{m}_{k-3}^b - r_{k-3}. \end{aligned}$$

Thus, $\mathbf{m}_{k+1}^b - \mathbf{m}_{k+1}^a = -r_{k-3} + \max((r_k + r_{k-2}), 2r_{k-1}) = r_{k+1}$. \square

Theorem 6. For all $n > 4$, the polynomials c_i defined by (49) satisfy the Somos-4 recurrence

$$c_{n-2}c_{n+2} = \alpha_n^\tau c_{n+1}c_{n-1} + \beta_n^\tau c_n^2, \quad (55)$$

with periodic coefficients,

$$\alpha_n^\tau = \alpha \left(\prod_{i=1}^4 \tau_i^{p_{n-i \bmod 8}} \right) \quad (56)$$

and

$$\beta_n^\tau = \beta \left(\prod_{i=1}^4 \tau_i^{q_{n-i \bmod 8}} \right), \quad (57)$$

with p and q as given in Corollary 2, and initial values $\{c_i = 1\}_{i=1}^4$.

Proof. From (49), (50) and the initial values we obtain

$$c_5 = \alpha c_2 c_4 - \beta c_3, \quad c_6 = \alpha c_3 c_5 + \beta c_1 c_4, \quad c_7 = \alpha c_1 c_4 c_6 + \beta c_2 c_5, \quad \text{and } c_8 = \alpha c_1 c_4 c_6 + \beta c_2 c_5, \quad (58)$$

Using Lemma 5, we find $a_n = g_n c_n$ and $b_n = (\prod_{i=1}^4 c_i^{r_{n-i+1}}) g_n$. Substituting these expressions in (45) gives, for $n > 8$

$$c_{n+2}g_{n+2} = g_{n-1}g_{n-2}g_n^2g_{n+1}(\alpha c_{n+1}c_{n-1} \prod_{i=1}^4 c_i^{2r_{n-i+1}+r_{n-i-1}} + \beta c_n^2 \prod_{i=1}^4 c_i^{r_{n-i+2}+r_{n-i-1}+r_{n-i}}).$$

From (46), we find:

$$\prod_{i=1}^4 c_i^{r_{n-i+3}} g_{n+2} = g_{n-1}g_{n-2}g_n^2g_{n+1}(\prod_{i=1}^4 c_i^{r_{n-i+2}+2r_{n-i+1}+r_{n-i}})c_{n-2}.$$

Eliminating g_{n+2} from the above yields

$$c_{n+2}c_{n-2} = \alpha \left(\prod_{i=1}^4 c_i^{r_{n-i+3}-r_{n-i+2}-r_{n-i}+r_{n-i-1}} \right) c_{n+1}c_{n-1} + \beta \left(\prod_{i=1}^4 c_i^{r_{n-i+3}-2r_{n-i+1}+r_{n-i-1}} \right) c_n^2,$$

which can be expressed in terms of p and q

$$r_{n-i+3} - r_{n-i+2} - r_{n-i} + r_{n-i-1} = x_{n-i+1} + x_{n-i} + x_{n-i-1} = p_{n-i \bmod 8},$$

and

$$r_{n-i+3} - 2r_{n-i+1} + r_{n-i-1} = x_{n-i+1} + 2x_{n-i} + x_{n-i-1} = q_{n-i \bmod 8}.$$

Taking unit initial values, the relations (58) are generated by (55). \square

3.2 Periodic Somos-5

For Somos-5 we follow the same steps. Homogenising $\sigma_n = a_n/b_n$ gives

$$a_{n+3} = w_{n+3} b_{n-2}, \quad (59)$$

$$b_{n+3} = b_{n+2} b_{n-1} b_n b_{n+1} a_{n-2}, \quad (60)$$

where $w_{n+3} := \gamma a_{n+2} a_{n-1} b_n b_{n+1} + \delta a_n a_{n+1} b_{n+2} b_{n-1}$. We take $\{b_i = 1\}_{i=1}^5$ and so $\{a_i = \sigma_i\}_{i=1}^5$. Iterating (59) and (60) five more times, we find:

$$\begin{aligned} a_{n+4} &= b_{n+2} b_{n+1} b_{n-1} q_1, \quad b_{n+4} = a_{n-2} a_{n-1} b_{n-1} b_n^2 b_{n+1}^2 b_{n+2}^2, \\ a_{n+5} &= a_{n-2} b_{n-1} b_n^2 b_{n+1}^2 b_{n+2}^2 q_2, \quad b_{n+5} = a_{n-2}^2 a_{n-1} a_n b_{n-1}^3 b_{n+1}^4 b_{n+2}^4, \\ a_{n+6} &= a_{n-2}^2 a_{n-1} b_{n-1}^3 b_{n+1}^6 b_{n+2}^5 q_3, \quad b_{n+6} = a_{n-2}^4 a_{n-1}^2 a_n a_{n+1} b_{n-1}^4 b_n^6 b_{n+1}^7 b_{n+2}^8, \\ a_{n+7} &= a_{n-2}^5 a_{n-1}^2 a_n b_{n-1}^6 b_{n+1}^{11} b_{n+2}^{12} q_4, \quad b_{n+7} = a_{n-2}^8 a_{n-1}^4 a_n^2 a_{n+1} a_{n+2} b_{n-1}^{12} b_{n+1}^{14} b_{n+2}^{15}, \\ a_{n+8} &= a_{n-2}^{12} a_{n-1}^5 a_n^2 a_{n+1} b_{n-1}^{14} b_n^{18} b_{n+1}^{24} b_{n+2}^{25} w_{n+3} q_5, \quad b_{n+8} = a_{n-2}^{15} a_{n-1}^8 a_n^4 a_{n+1}^2 a_{n+2} b_{n-1}^{15} b_n^{23} b_{n+1}^{27} b_{n+2}^{29} w_{n+3}, \end{aligned}$$

where $\{q_i\}_{i=1}^5$ are irreducible polynomials in $\{a_{n+i}, b_{n+i}\}_{i=-2}^2$, δ and γ . From (59) and (60), the system that gives a lower bound for multiplicities is:

$$\mathbf{m}_{n+3}^a = \min(\mathbf{m}_{n+2}^a + \mathbf{m}_{n-1}^a + \mathbf{m}_n^b + \mathbf{m}_{n+1}^b + \mathbf{m}_{n-2}^b; \mathbf{m}_n^a + \mathbf{m}_{n+1}^a + \mathbf{m}_{n+2}^b + \mathbf{m}_{n-1}^b + \mathbf{m}_{n-2}^b), \quad (61)$$

$$\mathbf{m}_{n+3}^b = \mathbf{m}_{n+2}^b + \mathbf{m}_{n-1}^b + \mathbf{m}_n^b + \mathbf{m}_{n+1}^b + \mathbf{m}_{n-2}^a. \quad (62)$$

To obtain a lower bound for $\mathbf{m}_n^a(w_k)$ and $\mathbf{m}_n^b(w_k)$, we solve (61) and (62) with the following initial values: $\mathbf{m}_{k+i}^a = \mathbf{m}_{k+i}^b = 0$ for all $i \in \{1, 2, 3, 4\}$ and $\mathbf{m}_{k+5}^a = \mathbf{m}_{k+5}^b = 1$. We find, for all $n \geq k+1$,

$$\mathbf{m}_n^a(w_k) = \mathbf{m}_n^b(w_k) = \mathbf{m}_{n-k},$$

where $\mathbf{m}_{n+4} = \mathbf{m}_{n+3} + \mathbf{m}_{n+2} + \mathbf{m}_{n+1} + \mathbf{m}_n + \mathbf{m}_{n-1}$ and $\mathbf{m}_1 = \mathbf{m}_2 = \mathbf{m}_3 = \mathbf{m}_4 = \mathbf{m}_5 - 1 = 0$. Then, the formulae for a_n and b_n in terms of a new sequence $\{d_k\}_{k=1}^\infty$ are given as follows, with $n > 5$,

$$a_n = \left(\prod_{i=1}^5 d_i^{\mathbf{m}_n^a(d_i)} \right) \left(\prod_{i=6}^{n-1} d_i^{\mathbf{m}_{n-i}^a} \right) d_n, \quad b_n = \left(\prod_{i=1}^5 d_i^{\mathbf{m}_n^b(d_i)} \right) \prod_{i=6}^{n-1} d_i^{\mathbf{m}_{n-i}^b}, \quad (63)$$

where sequences $\{\mathbf{m}_n^a(d_{i \leq 5})\}$ and $\{\mathbf{m}_n^b(d_{i \leq 5})\}$ are defined by (61) and (62) and the initial values $\{\mathbf{m}_j^a(d_i) = \delta_{ij}, \mathbf{m}_j^b(d_i) = 0\}_{i,j=1}^5$. These formulae clearly show that Somos-5 possesses the Laurent property. The differences between the multiplicities of $c_{i \leq 5}$ can be expressed in terms of the ultra-discrete Somos-5 sequence defined by

$$t_{n+5} = -t_n + \max(t_{n+4} + t_{n+1}, t_{n+3} + t_{n+2}), \quad (64)$$

and initial values $t_1 = -1, \{t_i = 0\}_{i=2}^5$. The quantity

$$y_{k+1} = t_{k+3} - t_{k+2} - t_{k+1} + t_k, \quad (65)$$

which relates to (28), satisfies the ultra-discrete QRT-map (related to 26),

$$y_{k+2} + y_{k+1} + y_k = \max(y_{k+1}, 0),$$

and is periodic of order 7. It follows from (61) and (62) that

$$\mathbf{m}_n^b(d_i) - \mathbf{m}_n^a(d_i) = t_{n-i+1}, \quad (66)$$

for all $1 \leq i \leq 5$.

Theorem 7. Let r, s be given as in (36). For all $n > 7$, the polynomials $d_{n>5}$ defined by (63), satisfy the Somos-5 recurrence with periodic coefficients

$$d_{n+3}d_{n-2} = \gamma_n d_{n+2}d_{n-1} + \delta_n d_n d_{n+1}, \quad (67)$$

where

$$\gamma_n = \gamma \left(\prod_{i=1}^5 \sigma_i^{r_{n-i \bmod 7}} \right), \quad \delta_n = \delta \left(\prod_{i=1}^5 \sigma_i^{s_{n-i \bmod 7}} \right), \quad (68)$$

and initial values $\{d_i = 1\}_{i=1}^5$.

Proof. Using (59), (60), initial values and (63), we find

$$\begin{aligned} d_6 &= \alpha \sigma_5 \sigma_2 + \beta \sigma_3 \sigma_4, & d_7 &= \alpha d_6 \sigma_3 + \beta \sigma_4 \sigma_5 \sigma_1, & d_8 &= \alpha d_7 \sigma_4 + \beta \sigma_2 d_6 \sigma_5, \\ d_9 &= \alpha d_8 \sigma_5 \sigma_1 + \beta \sigma_3 d_7 d_6, & \text{and } d_{10} &= \alpha d_9 d_6 \sigma_2 + \beta \sigma_1 \sigma_4 d_8 d_7. \end{aligned} \quad (69)$$

For all $n > 10$, from (63) and (66), we find $a_n = g_n d_n$ and $b_n = (\prod_{i=1}^5 d_i^{t_{n-i+1}}) g_n$. Substitution in equation (59) gives

$$\begin{aligned} d_{n+3}g_{n+3} &= g_{n-2}g_{n-1}g_n g_{n+1}g_{n+2} (\gamma d_{n+2}c_{n-1} \prod_{i=1}^5 d_i^{t_{n-i+1}+t_{n-i+2}+t_{n-i-1}} \\ &\quad + \delta d_n d_{n+1} \prod_{i=1}^5 d_i^{t_{n-i+3}+t_{n-i}+t_{n-i-1}}). \end{aligned}$$

From (60), we find:

$$g_{n+3} \prod_{i=1}^5 d_i^{t_{n-i+4}} = g_{n-2}g_{n-1}g_n g_{n+1}g_{n+2} (\prod_{i=1}^5 d_i^{t_{n-i+3}+t_{n-i}+t_{n-i+1}+t_{n-i+2}}) d_{n-2}.$$

Eliminating g_{n+3} from the above two equations yields

$$\begin{aligned} d_{n+3}d_{n-2} &= \gamma \left(\prod_{i=1}^5 d_i^{t_{n-i+4}-t_{n-i+3}-t_{n-i}+t_{n-i-1}} \right) d_{n+2}d_{n-1} \\ &\quad + \delta \left(\prod_{i=1}^5 d_i^{t_{n-i+4}-t_{n-i+1}-t_{n-i+2}+t_{n-i-1}} \right) d_n d_{n+1}, \end{aligned}$$

which can be expressed in terms of r and s as follows

$$t_{n-i+4} - t_{n-i+3} - t_{n-i} + t_{n-i-1} = y_{n-i+1} + y_{n-i-1} = r_{n-i \bmod 7},$$

and

$$t_{n-i+4} - t_{n-i+1} - t_{n-i+2} + t_{n-i-1} = y_{n-i+1} + y_{n-i} + y_{n-i-1} = s_{n-i \bmod 7}.$$

Taking $\{d_i = 1\}_1^5$, then (69) are generated by (67). \square

3.3 On the Laurent property of periodic Somos-4 & 5 sequences

As the periodic Somos-4 & 5 sequences we have derived are special cases of equation (9) and the condition (10) is satisfied, they possess the Laurent property.

If we would not have had the Hirota-Miwa equation at hand, or one wants a direct proof this can be done. Actually, most of the work has been done already. Considering (55), the substitution $c_n = a_n/b_n$ yields the same system of recurrences (45), (46) for polynomials a_n and b_n . The only difference is that in the expression for w_{n+2} , the α and β are now periodic functions of n with

period 8. This means that the iteration of the recurrences (four more times) has to be repeated for different values of $n \equiv i \pmod{8}$, with $i \in \{0, 1, 2, \dots, 7\}$. For each value of i we found that w_{n+2} does not divide a_{n+k} or b_{n+k} , with $k = 3, 4, 5$, and that it does divide both a_{n+6} and b_{n+6} . As the system of recurrences is similar, the derived ultra-discrete system (47), (48) is the same, polynomials c_n are defined by equation (49), and the proof carries over. Also no surprises were found when iterating the system (29,30) five more times, for $p = 7$ different values for $n \pmod{p}$.

4 From DTKQ equations to sequences of polynomials which satisfy rational recurrences

The aim of this section is to study how the second and third order DTKQ equations give rise to recurrences that possess the Laurent property. The N th order DTKQ equation

$$\sum_{s=0}^N u_{n+s} \prod_{q=1}^{N-1} u_{n+q} = \alpha. \quad (70)$$

was derived in [7], through applying the principle of duality for difference equations, and it was shown to admit sufficiently many integrals to be completely integrable. The growth of the equations has been studied in [22].

4.1 From the second order DTKQ equation to a fifth order Laurent recurrence with four terms.

In the case $N = 2$, the DTKQ equation is

$$u_{n+2} = \frac{\alpha}{u_{n+1}} - u_n - u_{n+1}. \quad (71)$$

Substituting $u_n = a_n/b_n$ in (71) and identifying the numerators and denominators, we get a system of recurrences for polynomial sequences $\{a_n\}$ and $\{b_n\}$:

$$a_{n+2} = \alpha b_n b_{n+1}^2 - a_n a_{n+1} b_{n+1} - b_n a_{n+1}^2, \quad (72)$$

$$b_{n+2} = a_{n+1} b_n b_{n+1}, \quad (73)$$

with $a_1 = u_1$, $a_2 = u_2$, $b_1 = b_2 = 1$. Therefore, a_n and b_n are polynomials in the variables u_1 and u_2 . Via recursive factorisation, we find, see [22], in terms of a polynomial sequence $\{e_n\}$,

$$a_n = \begin{cases} e_n & \text{if } n \leq 3, \\ e_n e_{n-3} \prod_{i=2}^{n-3} e_i^{\mathfrak{m}_{n-i-2}} & \text{if } n > 3, \end{cases} \quad (74)$$

$$b_n = \begin{cases} 1 & \text{if } n \leq 2, \\ e_2 & \text{if } n = 3, \\ e_{n-1} e_{n-2} \prod_{i=2}^{n-3} e_i^{\mathfrak{m}_{n-i-2}} & \text{if } n > 3, \end{cases} \quad (75)$$

$$(76)$$

with $\mathfrak{m}_1 = 2$, $\mathfrak{m}_2 = 6$, and $\mathfrak{m}_l = 2\mathfrak{m}_{l-1} + \mathfrak{m}_{l-2}$.

Theorem 8. *The polynomials $e_{n>3}$ satisfy*

$$\frac{e_{n-1}e_{n-5}}{e_{n-3}^2} + \frac{e_{n-1}^2 e_{n-4}^2}{e_{n-3}^2 e_{n-2}^2} + \frac{e_{n-4}e_n}{e_{n-2}^2} = \alpha, \quad (77)$$

where

$$\{e_i = 1\}_{i=-1}^1, \quad e_2 = u_2 \quad \text{and} \quad e_3 = \alpha - u_1 u_2 - u_2^2. \quad (78)$$

We remark that a reduction of order, by introducing the variable e_{n+1}/e_{n-1} , is apparent, however, this does not preserve Laurentness. Furthermore we should mention that the fact that the rational recurrence (77) with initial values (78) produces a polynomial sequence does not follow from the Laurent property of (77). One needs a strong Laurent property such as given in [28] for Somos sequences.

Proof. From (74), (72), (73) and initial values, we obtain

$$e_2 = a_2, \quad e_3 = a_3 = \alpha b_1 b_2^2 - a_1 a_2 b_2 - b_1 a_2^2 = \alpha - u_1 u_2 - u_2^2.$$

Similarly, we find

$$e_4 = a_4 = \alpha u_2^2 - e_3 u_2^2 - e_3^3, \quad (79)$$

$$e_5 = \frac{a_5}{e_2 g_5} = \frac{\alpha u_2^2 e_3^2 - e_4 e_3^2 - e_4^2}{u_2^2}, \quad (80)$$

$$e_6 = \frac{a_6}{e_3 g_6} = \frac{\alpha e_3^2 e_4^2 - e_5 e_4^2 - e_5^2 u_2^2}{u_2 e_3^2}. \quad (81)$$

Now consider, for $n > 4$, replacing a_{n+i} by $e_{n+i} c_{n-3+i} g_{n+i}$ and b_{n+i} by $e_{n-1+i} e_{n-2+i} g_{n+i}$ in the right hand side of equation (72):

$$e_{n+2} e_{n-1} g_{n+2} = g_n g_{n+1}^2 e_{n-1} e_{n-2} (\alpha e_n^2 e_{n-1}^2 - e_n^2 e_{n-3} e_{n+1} - e_{n+1}^2 e_{n-2}^2).$$

From equation (73) we find $g_n g_{n+1}^2 = \frac{g_{n+2}}{e_{n-1}^2 e_{n-2}^2}$ and these combine to give the recurrence equation for e 's, (77). By taking $e_{-1} = e_0 = e_1 = 1$, the recurrence equation (77) generates the above expressions (79), (80) and (81) \square

We could now recursively factorize the equation (77), but if one just wants to verify the Laurent property there is an easier method, as described in [38]. By iterating the map five times we obtain $\{e_n = p_n/q_n\}_{n=5}^{10}$, for polynomials p_n and monomials q_n in the initial values $\{e_n\}_{n=1}^5$. As p_5 is prime to q_5 for all $n \in \{6, 7, 8, 9, 10\}$ the recurrence (77) satisfies the Laurent property.

4.2 From the third order DTKQ equation to a sixth order Laurent recurrence with five terms, with coefficients that are periodic with period 8.

Taking $N = 3$ in equation (70), this gives the third order DTKQ equation,

$$u_{n+3} = \frac{\alpha}{u_{n+1} u_{n+2}} - u_n - u_{n+1} - u_{n+2},$$

and homogenising yields,

$$a_{n+3} = \alpha b_{n+1}^2 b_{n+2}^2 b_n - a_{n+1} a_{n+2} a_n b_{n+1} b_{n+2} - a_{n+1}^2 a_{n+2} b_{n+2} b_n - a_{n+1} a_{n+1}^2 b_{n+1} b_n, \quad (82)$$

$$b_{n+3} = b_{n+1} b_{n+2} b_n a_{n+1} a_{n+2}. \quad (83)$$

If we choose $\{a_i = u_i, b_i = 1\}_{i=1}^3$ then a_i and b_i are polynomials in the initial variables u_1, u_2, u_3 and parameter α . A sequence of polynomials $\{z_i\}$ is defined by:

$$a_n = \begin{cases} z_n & \text{if } n < 5, \\ z_3 z_5 & \text{if } n = 5, \\ z_2^{m_n^a(z_2)} \prod_{i=3}^{n-3} z_i^{m_{n-i-2}} z_{l-3} z_{l-2} z_l & \text{if } n > 5, \end{cases} \quad (84)$$

$$b_n = \begin{cases} 1 & \text{if } n < 4, \\ z_2 z_3 & \text{if } n = 4, \\ z_2^{m_n^b(z_2)} \prod_{i=3}^{n-3} z_i^{m_{n-i-2}} z_{n-2}^2 z_{n-1} & \text{if } n > 4, \end{cases} \quad (85)$$

where \mathbf{m} is the integer sequence defined by $\mathbf{m}_1 = 4$, $\mathbf{m}_2 = 13$, $\mathbf{m}_3 = 37$ and $\mathbf{m}_n = 2\mathbf{m}_{n-1} + 2\mathbf{m}_{n-2} + \mathbf{m}_{n-3}$. In this case the ultra-discretisation of the homogenised system does not give us a sharp bound on the multiplicities $m_n^a(z_2)$ and $m_n^b(z_2)$. By using prime numbers as initial values we were able to iterate the map (82,83) a little further than usual and thus observe the following.

Conjecture 9. *The difference of the multiplicities of z_2 in a_n and b_n is periodic, we have $m_n^a(z_2) - m_n^b(z_2) = \zeta_n \bmod 8$, with $\zeta = [0, 1, 0, -1, -1, 2, -1, -1]$.*

Assuming the conjecture, from (84,85), it follows that

$$\frac{a_n}{c_{n-3}c_n g_n} = z_2^{\max(0, \zeta_n \bmod 8)} \quad \text{and} \quad \frac{b_n}{c_{n-2}c_{n-1}g_n} = z_2^{\max(0, -\zeta_n \bmod 8)}$$

are polynomial sequences in z_2 . Using these functions we find the following theorem

Theorem 10. *The polynomials $z_{n>4}$, as defined by (82), satisfy*

$$\epsilon_n \frac{z_{n-3}z_{n+1}}{z_{n-1}^2} + \epsilon_{n+1} \frac{z_{n-2}^2 z_{n+1}^2}{z_{n-1}^2 z_n^2} + \epsilon_{n+2} \frac{z_{n-2}z_{n+2}}{z_n^2} + \epsilon_{n+3} \frac{z_{n-2}z_{n+3}}{z_{n-1}z_{n+2}} = \frac{\alpha}{\epsilon_{n+1}\epsilon_{n+2}} \frac{z_n z_{n+1}}{z_{n-1}z_{n+2}}, \quad (86)$$

with $\epsilon = u_2^{\zeta_n \bmod 8}$, $\{z_n = 1\}_{n=-1}^2$, $z_3 = u_3$ and $z_4 = \alpha - u_2 u_3 (u_1 + u_2 + u_3)$.

The Laurentness of (86) can be verified as before, this time the iteration has to be repeated for every congruence class $n \bmod 8$.

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